quently (from P. XVI.) is established the hypothesis of right angle; and at the same time (from P. XIII.) the Euclidean postulate.

Moreover I have supposed the side DF, or AH assumed equal to it, to be less than the side AC. For if it were equal, and so the point H should fall upon the point C, then the angle BCA would be equal (by hypothesis) to the angle EFD, or GCA (which then it would become) a part to the whole; which is absurd.

But if it were greater, and so the join GH should cut BC itself in some point, now the external angle ACB would be from the hypothesis equal (against Eu. I. 16) to the internal and opposite angle (which then would become) AHG, or GHA.

Therefore I have rightly supposed the side DF of one triangle to be less than the side AC of the other triangle, in accordance with the hypothesis now established.

Wherefore from any two triangles mutually equiangular, but not also mutually equilateral, the Euclidean postulate is established.

Quod intendebatur.

[To be Continued.]

HISTORICAL SURVEY OF THE ATTEMPTS AT THE COMPUTATION AND CONSTRUCTION OF π.

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[NOTE. The following article is translated (by permission) from Professor Klein's recent work, *Vortraege ueber ausgewaehlte Fragen der Elementargeometrie, ausgearbeitet von F. Taegert*, Leipzig, Teubner, 1895. The work can not be too highly commended to teachers, since it is one of those exceedingly rare treatises in which a master of modern mathematics has treated elementary subjects from his high point of view.
Michigan State Normal School, December, 1895.]

Later in this work it will be proved that the number π belongs to that class of numbers known as transcendent, whose existence was shown in the preceding chapter. This fact was first proved by Lindemann in 1882, and a problem was thereupon settled which, so far as our information extends, has occupied the attention of mathematicians for 4000 years, namely, that of the quadrature of the circle.

It is evident that if the number π is not algebraic it cannot be constructed by means of the compasses and ruler. Hence the quadrature of the circle is, in the sense understood by the ancients, impossible. It is of greatest interest to follow the fortunes of this problem in the various epochs of Science, as ever new attempts were made to find a solution by means of the ruler and the

compasses, and to see how these necessarily fruitless attempts nevertheless worked for advancement in the manifold realm of mathematics.

The following brief historical survey is based upon Rudio's excellent treatise, *Archimedes, Huygens, Lambert, Legendre; Vier Albandlungen ueber die Kreismessung*, *Leipzig*, 1892. In this work are given in German translation the contributions of the writers named. Even though the presentation of the matter is remote from the more modern methods here discussed,* nevertheless it includes many very interesting details which are of especial value in elementary teaching.

1. Among the attempts to determine the ratio of the diameter to the circumference we may first distinguish the empirical stage in which it was sought to attain the desired end through measuring or estimating. The oldest known mathematical work, the Rhind Papyrus (c. 2000 B. C.) contains the problem in the well-known form, to transform a circle into a square of equal area. The writer of the papyrus, Ahmes, lays down the following rule: Cut off $\frac{1}{9}$ of a diameter and construct a square on the remainder; this has the same area as the circle. The value of π thus obtained is $(\frac{16}{9})^2=3.16\ldots\ldots$, not very inexact. Still farther from the correct value is that of $\pi=3$ which is found in the Bible. (I Kings, 7 : 23, and II Chron. 4 : 2.)

2. The Greeks raised themselves above this impirical standpoint, and especially Archimedes, who in his work *Κύκλου μέτρησις* computes the area of the circle by the help of inscribed and circumscribed polygons, as is still done in the schools. His method remained in use until the invention of the differential calculus, and was extended and made practically usable especially by Huygens (†1654) in his work *De circuli magnitudine inventa.*

As in the case of the duplication of the cube and the trisection of an angle the Greeks then sought to attain the quadrature of the circle by the help of higher curves.

We may, for example, consider the curve, $y=\text{arc}\sin x$ [usually written in English $y=\sin^{-1}x$; the Continental form will be followed in this translation] which represents the curve of sines placed vertically. Geometrically, π appears as a special ordinate of this curve, analytically as a special value of our transcendent function. Apparatus which describes transcendent curves we will call transcendent apparatus. A piece of transcendent apparatus which draws the curve of sines gives us a real construction for π. The curve $y=\text{arc}\sin x$ we designate now-a-days as an *integral curve*, because it can be defined by means of the integral of an algebraic function: $y=\int\frac{dx}{\sqrt{1-x^2}}$. The ancients called such a curve a Quadratrix or *τετραγωνίζουσα*. The best known of these is the Quadratrix of Dinostratus (c. 350 B. C.) which, however, had been already constructed by Hippias of Elis (c. 420 B. C.) for the trisection of an angle. It may be geometrically defined as follows: On the line OB and the arc AB two points, M and L, move

*In a note to the translator Professor Klein says: "This remark concerning Rudio's work is not happily expressed. The meaning is not that modern researches, so far as then carried, are not given in the work, but they are not deduced."

with uniform velocity. They start at the same time from O and A, respectively, and they reach B at the same time. If OL is drawn, and through M the parallel to OA which meets OL at P, then P is a point of the Quadratrix. From this definition it follows that y and θ are proportional. Further, since for $y=1$, $\theta=\frac{\pi}{2}$, we have $\theta=\frac{\pi}{2}y$, and from $\theta=\text{arc tan}\frac{y}{x}$ the equation of the curve becomes $\frac{y}{x}=\tan\frac{\pi}{2}y$. The point in which the line cuts the x-axis will be found from $x=\frac{y}{\tan\frac{\pi}{2}y}$ if y becomes 0. Since for small values the tangent equals its argument, it follows that $x=\frac{2}{\pi}$. Hence the radius of the circle is the mean proportional between the quadrant of the circle and the abscissa of the point of intersection of the Quadratrix with the x-axis. The Quadratrix can, therefore, be used in the rectification problem, and hence for the quadrature of the circle. Fundamentally, however, the curve is only a geometric formulation of the rectification problem, that is so long as no apparatus is given by which it can be described by a continuous line.

3. The rise of modern analysis occurs in the period from 1670 to 1770, a period characterized by the names of Leibniz, Newton, and Euler. In the midst of so many great discoveries following closely on one another, it is natural that strict criticism took a somewhat backward step. Among these discoveries is one of especial concern to us, the development of the theory of series. Especially for π were a great number of approximations brought forward, of which we may mention only the so-called Leibniz series (which, however, was known before Leibniz): $\frac{\pi}{4}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots\ldots$ Furthermore this period brings the discovery of the connection between e and π. The number e and the natural logarithms and with them the exponential function are first found *in embryo* in the works of Napier (1614). This number seemed at first to have no relation to the circular functions and to the number π, until Euler had the courage to attack the problem by means of imaginary exponents. In this way he reached the celebrated formula $e^{ix}=\cos x+i\sin x$, which for $x=\pi$ becomes $e^{i\pi}=-1$. This formula is without doubt one of the most notable of all mathematics. With it are connected the modern proofs of the transcendence of π since they first show the transcendence of e.

4. After 1770 criticism again took the upper hand. In 1770 appeared Lambert's work, *Vorläufige Kentnisse für die, so die Quadratur des Cirkuls suchen.* He treated there and elsewhere the irrationality of π. In 1794 Legendre showed conclusively in his *Éléments de Géométrie* that π and π^2 are irrational numbers.

5. But it was not until a hundred years later than this that modern research began. The starting point of this research is the work of Hermite, *Sur la fonction exponentielle* (*Compt. Rend.* 1873, published separately in 1874). In this

is proved the transcendence of e. Closely following Hermite came the same proof for π by Lindemann in a dissertation *Ueber die Zahl* π (*Math. Ann.* 20, 1882. See also the proceedings of the Berlin and Paris academies). With this the matter was now for the first time settled, nevertheless the treatment given by Hermite and Lindemann is very complicated.

The first simplification was given by Weierstrass in the *Berliner Berichte* in 1885. The above mentioned works Bachman embodied in his text-book, *Vorlesungen ueber die Natur der Irrationalzahlen*, 1892.

The spring of 1893 brought, however, new and very important simplifications. In the first rank should be named the developments of Hilbert in the *Göttinger Nachrichten.* Hilbert's proof is not wholly elementary; it contains still a remnant of Hermite's course of reasoning in the integral

$$\int_0^{\infty} z^{\rho} e^{-z} dz = \rho\,!.$$

But Herwitz and Gordan soon after showed that this transcendental part might be eliminated. (*Göttinger Nachrichten* and *Comptes Rendus* respectively; all three dissertations are reproduced in the *Math. Annalen*, Bd. 43, either literally or somewhat extended). So the matter has now become so elementary that it is generally available.

INTRODUCTION TO SUBSTITUTION GROUPS.

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[Continued from November Number.]

The Construction of Non-Primitive Groups with Two Systems of Non-Primitivity.

Let the degree of the required non-primitive group be $2n$, and consider the $(n\,!)^2$ substitutions

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all}\ a_1 b_1 . a_2 b_2 \ldots\ldots a_n b_n$$

and also the group of order $2(n\,!)^2$

$$(a_1 a_2 \ldots\ldots a_n)\text{all}(b_1 b_2 \ldots\ldots b_n)\text{all}(a_1 b_1 . a_2 b_2 \ldots\ldots a_n b_n).$$

The latter is clearly a non-primitive group of degree $2n$ and the former are the substitutions of this group which interchange the systems. It is easily seen